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Tuneshifts, Tunespreads and Decoherence

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In a ring with amplitude dependent tunes, particles subjected to the same dipole kick at some initial time will subsequently decohere. Several formalisms are used to calculate the decoherence time for a particular model Hamiltonian, and they are all shown to yield the same result. Because the precise definition of the decoherence time is somewhat arbitrary, alternative definitions are briefly treated, to show the changes they would make to the results presented here. A more general model Hamiltonian is also briefly studied. The calculation of the decoherence factor, which gives the time evolution of the beam centroid after the kick, not just the overall decoherence time, is also studied, and heuristic derivations for certain special cases are given. The validity of some of the approximations involved in the derivations is investigated, and an approximate bound is given.

1 Introduction

In general, the various particle trajectories in a storage ring have different tunes, or phase advances per unit time. This means that, if a beam is given a dipole kick, to study the response to some stimulus, the individual particles will eventually get out of phase with each other, even though they were all kicked by the same amount. This phenomenon is called “decoherence.” It results, in particular, in a modulation of the beam centroid motion. If the dynamics were linear, the centroid would execute a pure betatron oscillation, but because of decoherence its amplitude varies with time. Depending on the nature of the decoherence mechanism, the beam centroid amplitude may or may not decay irreversibly to zero [1].

There are many ways to calculate the decoherence time. First, though, it should be noted that decoherence is a statistical phenomenon — it arises because of the loss of phase coherence of a *distribution* of particles — and so the “decoherence time” is a *qualitative* concept. The decoherence criterion used in this report will be defined below, and alternative criteria will be discussed later. In this report, I shall calculate the decoherence time, using a particular Hamiltonian, using several different methods. Partly, this work proves that all of these methods are equivalent, and partly it explains how to apply these different methods to the same problem. For lack of standard names, I have christened the formalisms myself: the “Edwards method,” “differential equation method,” “canonical transform method,” “Deprit’s/Michelotti’s method,” and “Forest’s/Yokoya’s method.” *I do not claim that any one method is superior to any other. I say only that the methods are equivalent, and prove it by deriving the same answer (the decoherence time) from all of them.* I also consider, in addition, the generalization of the simple model Hamiltonian used throughout most of the report to a model of an arbitrary distribution of sextupoles (a “discrete-sextupole Hamiltonian”). The calculations in this report are purely theoretical. Experimental data have been taken by the E778 collaboration at the Fermilab Tevatron, and details of the comparison between theory and experiment will be given elsewhere [2].

In addition to the decoherence time, there is also the concept of the “decoherence factor.” This is the function that shows how the beam centroid amplitude *varies* with time, as opposed to merely giving the decoherence *time*. I investigate some details of the validity of the approximations used in calculating the decoherence factor for a specific model [1], and study the derivation of the decoherence time from their solution, for various circumstances. I show that the results of Ref. [1] agree with “simple-minded” heuristic derivations, and offer some comments on some special cases.

2 General Remarks

2.1 Hamiltonian

The Hamiltonian used is

$$H = I\delta + kI^{3/2} \cos(3\psi) . \quad (1)$$

Here I and ψ are the action-angle variables obtained by diagonalizing the linear dynamics, $\delta = \nu - p/3$ is the difference between the “unperturbed tune” ν and the nearest resonant value $p/3$, where p is an integer not divisible by 3. The quantity k is a constant.

Let us call the independent variable θ . Then the equations of motion are

$$\frac{dI}{d\theta} = \{I, H\} = -\frac{\partial H}{\partial \psi} = 3kI^{3/2} \sin(3\psi) \quad (2)$$

and

$$\frac{d\psi}{d\theta} = \{\psi, H\} = \frac{\partial H}{\partial I} = \delta + \frac{3}{2}kI^{1/2} \cos(3\psi) . \quad (3)$$

2.2 Decoherence criterion

Without nonlinearities, $\psi \equiv \psi_0 = \theta\delta$. With nonlinearities,

$$\psi \equiv \psi_{nl} = \theta\delta' + \text{oscillatory terms} , \quad (4)$$

where δ' is a constant not equal to δ . I shall neglect the oscillatory terms for now because they average to zero. I shall check the validity of this approximation below. Then, after N turns, $\psi_0 = 2\pi\delta N$. To obtain the decoherence time we define N such that, after N turns, $\psi_{nl} = \psi_0 \pm 2\pi$. The \pm sign is chosen so that N is positive, and depends on the sign of the tuneshift, which is negative for a sextupole induced resonance. Then

$$\begin{aligned} \psi_{nl} &= \psi_0 \pm 2\pi \\ 2\pi\delta'N &= 2\pi\delta N \pm 2\pi \\ N &= \frac{1}{|\delta - \delta'|} . \end{aligned} \quad (5)$$

The above criterion is really only applicable for small kicks to the beam, i.e. the displacement of the beam centroid from the closed orbit should be much less than the transverse beam size/standard deviation $\langle x^2 \rangle$. This is because the “reference particle” in the above criterion, which should really be at the beam centroid, *is on the closed orbit*. If the beam centroid

displacement much exceeds the beam size, then the decoherence criterion will change. It will then include the initial amplitude of the kick. To see this, let us assume that the tune depends on amplitude via

$$\delta' = \delta_0 + \delta_2 I, \quad (6)$$

where δ_0 and δ_2 are constants, and the initial kick has an amplitude $|I_{\text{centroid}}| = \lambda^2 \sigma^2$, with $\lambda \gg 1$ and $\sigma^2 = \langle I \rangle$ as the transverse emittance. Then we take ψ_0 to refer to the beam centroid

$$\psi_0 = 2\pi[\delta_0 + \delta_2 \lambda^2 \sigma^2]N \quad (7)$$

and we evaluate ψ_{nl} on a trajectory roughly one standard deviation away, viz.

$$\psi_{nl} \simeq 2\pi[\delta_0 + \delta_2(\lambda\sigma \pm \sigma)^2]N, \quad (8)$$

and so

$$\begin{aligned} \psi_{nl} - \psi_0 &= \pm 2\pi \\ N\delta_2\sigma^2[(\lambda \pm 1)^2 - \lambda^2] &\simeq \pm 1 \\ N^{-1} &\simeq 2\lambda|\delta_2|\sigma^2 = 2\lambda\langle\Delta\nu\rangle. \end{aligned} \quad (9)$$

I prefer the notation $\langle\Delta\nu\rangle$ to $\langle\Delta\delta\rangle$ for the tunespread. The decoherence time is shorter, by a factor of roughly λ . The factor of 2 can be discounted because it will change if a more accurate average over the particle distribution is performed. We shall see an explicit example of these ideas below, when we examine the results of Ref. [1].

2.3 Expansion parameter

The calculations below will all employ perturbation theory. The expansion will sometimes be formulated in powers of k , and sometimes in powers of $I^{1/2}$. Since neither of these are dimensionless quantities, one must really be more precise: what is the expansion parameter?

Note, from the Hamiltonian in Eq. (1), that $kI^{1/2}$ is dimensionless, and, from the calculations below, we shall see that this is really the expansion parameter. In practice, however, I is not the same for all particles, and so, for numerical work, a more reasonable expansion parameter is $k\sigma$, where σ^2 is the transverse emittance. Most of the time in the calculations below, this point will be glossed over, and the perturbation expansion will be expressed as a series in powers of k only. The reader must understand that this is just for brevity.

3 “Edwards Method”

I shall first consider a decoherence calculation used by Edwards [3], and I shall call it the “Edwards method” for lack of a better name. One rewrites Eq. (3) in the form

$$\frac{d\psi}{\delta + \frac{3}{2}kI^{1/2}\cos(3\psi)} = d\theta , \quad (10)$$

hence

$$\int_0^\psi \frac{d\psi'}{\delta + \frac{3}{2}kI^{1/2}(\psi')\cos(3\psi')} = \int_0^\theta d\theta' = \theta . \quad (11)$$

Now H is an invariant of the motion, so put $H = E = \text{constant}$, and solve Eq. (1) for $I = I(\psi)$. Thus

$$I(\psi)\delta + kI^{3/2}(\psi)\cos(3\psi) = E , \quad (12)$$

and substitute this into Eq. (11). Edwards integrated Eq. (11) numerically. To proceed further analytically, I shall solve for $I = I(\psi)$ assuming k is small, and expand the integrand of Eq. (11) in powers of k . Thus I expand $I = I_0 + I_1 + \dots$, where $I_n \propto k^n$. The first approximation, using Eq. (12), is $I_0 = E/\delta$. The second approximation is given by

$$\begin{aligned} (I_0 + I_1 + \dots)\delta + k(I_0 + I_1 + \dots)^{3/2}\cos(3\psi) &= E \\ I_1\delta + kI_0^{3/2}\cos(3\psi) &= 0 \\ I_1 &= -\frac{k}{\delta}\left(\frac{E}{\delta}\right)^{3/2}\cos(3\psi) . \end{aligned} \quad (13)$$

Substituting into Eq. (11),

$$\begin{aligned} \left[\delta + \frac{3}{2}kI^{1/2}(\psi)\cos(3\psi)\right]^{-1} &\simeq \left[\delta + \frac{3}{2}k\left(\frac{E}{\delta} - \frac{k}{\delta}\left(\frac{E}{\delta}\right)^{3/2}\cos(3\psi)\right)^{1/2}\cos(3\psi)\right]^{-1} \\ &= \left[\delta + \frac{3}{2}k\left(\frac{E}{\delta}\right)^{1/2}\left(1 - \frac{k}{\delta}\left(\frac{E}{\delta}\right)^{1/2}\cos(3\psi)\right)^{1/2}\cos(3\psi)\right]^{-1} \\ &\simeq \left[\delta + \frac{3}{2}k\left(\frac{E}{\delta}\right)^{1/2}\left(1 - \frac{1}{2}\frac{k}{\delta}\left(\frac{E}{\delta}\right)^{1/2}\cos(3\psi)\right)\cos(3\psi)\right]^{-1} \\ &= \left[\delta + \frac{3}{2}k\left(\frac{E}{\delta}\right)^{1/2}\cos(3\psi) - \frac{3}{4}\frac{k^2}{\delta}\left(\frac{E}{\delta}\right)\cos^2(3\psi)\right]^{-1} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\delta} \left[1 + \frac{3k}{2\delta} \left(\frac{E}{\delta} \right)^{1/2} \cos(3\psi) - \frac{3k^2}{4\delta^2} \left(\frac{E}{\delta} \right) \cos^2(3\psi) \right]^{-1} \\
&= \frac{1}{\delta} \left[1 - \frac{3k}{2\delta} \left(\frac{E}{\delta} \right)^{1/2} \cos(3\psi) + \frac{3k^2}{4\delta^2} \left(\frac{E}{\delta} \right) \cos^2(3\psi) \right. \\
&\quad \left. + \frac{9k^2}{4\delta^2} \left(\frac{E}{\delta} \right) \cos^2(3\psi) + \dots \right] \\
&= \frac{1}{\delta} \left[1 - \frac{3k}{2\delta} \left(\frac{E}{\delta} \right)^{1/2} \cos(3\psi) + 3 \frac{k^2 E}{\delta^3} \cos^2(3\psi) + \dots \right] . \quad (14)
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^\psi \frac{d\psi'}{\delta + \frac{3}{2}kI^{1/2}(\psi') \cos(3\psi')} \\
&\simeq \int_0^\psi \frac{d\psi'}{\delta} \left[1 - \frac{3k}{2\delta} \left(\frac{E}{\delta} \right)^{1/2} \cos(3\psi) + 3 \frac{k^2 E}{\delta^3} \cos^2(3\psi) + \dots \right] \\
&\simeq \frac{\psi}{\delta} \left(1 + \frac{3k^2 E}{2\delta^3} \right) + \text{oscillatory terms} . \quad (15)
\end{aligned}$$

Substituting into Eq. (11),

$$\begin{aligned}
\frac{\psi}{\delta} \left(1 + \frac{3k^2 E}{2\delta^3} \right) &\simeq \theta \\
\psi &\simeq \theta \delta \left(1 - \frac{3k^2 E}{2\delta^3} \right) \\
&= \theta \left(\delta - \frac{3k^2 E}{2\delta^2} \right) , \quad (16)
\end{aligned}$$

and, using Eq. (5),

$$N = \frac{1}{|\delta - \delta'|} = \frac{2\delta^2}{3k^2 E} . \quad (17)$$

In Ref. [3] the decoherence time is calculated in a different way. One first finds n_1 and n_2 , for $k = k_1$ and $k = k_2$ respectively, via

$$n_{1,2}\delta = \int_0^{2\pi} \frac{d\psi'}{[\delta + \frac{3}{2}kI^{1/2}(\psi') \cos(3\psi')]_{E=E_1, E_2}} , \quad (18)$$

and then finds n such that $(n+1)n_1 = nn_2$, which yields

$$n = \frac{n_1}{|n_1 - n_2|} . \quad (19)$$

The decoherence time is given by $N = nn_1$. In terms of the formalism used in this note,

$$n_1\delta = \frac{\psi}{2\pi} , \quad n_2\delta = \frac{\psi}{2\pi} \left(1 + \frac{3k^2 E}{2\delta^3} \right) , \quad (20)$$

with $\psi = 2\pi$, $E_1 = 0$ and $E_2 = E$. Then

$$n = \frac{n_1}{|n_1 - n_2|} = \frac{\frac{1}{\delta}}{\frac{1}{\delta} \left(1 + \frac{3k^2 E}{2\delta^3} \right) - \frac{1}{\delta}} = \frac{2\delta^3}{3k^2 E} . \quad (21)$$

Then

$$N = nn_1 = \frac{2\delta^3}{3k^2 E} \frac{1}{\delta} = \frac{2\delta^2}{3k^2 E} , \quad (22)$$

which is the same as Eq. (17).

4 Numerical Results

Before turning to the other methods, let us check the above formula against the numerical results quoted in Ref. [3]. To do this, I need to relate the parameters used in the various calculations. The Hamiltonian of Ref. [3] is

$$H_1 \equiv \frac{a^3}{3} \frac{A}{4} \cos(3\psi) + \frac{a^2}{2} (2\pi\delta) . \quad (23)$$

I shall determine the relationship between the above symbols and the ones used in this report. First, the independent variable is not θ , but n , which is related to θ via $\theta = 2\pi n$. Therefore, to compare the above Hamiltonian with Eq. (1), one first divides by 2π :

$$H'_1 = \frac{a^2}{2} \delta + \frac{1}{2\pi} \frac{a^3}{3} \frac{A}{4} \cos(3\psi) . \quad (24)$$

Thus $I = a^2/2$, or $a = \sqrt{2I}$, and so

$$H'_1 = I\delta + \frac{1}{2\pi} \frac{(2I)^{3/2}}{3} \frac{A}{4} \cos(3\psi) = I\delta + \frac{\sqrt{2}}{12\pi} A I^{3/2} \cos(3\psi) . \quad (25)$$

Hence

$$k = \frac{\sqrt{2} A}{12\pi} . \quad (26)$$

A quantity $a_0 = 8\pi\delta/A$ is introduced, and the definition $a = ua_0$, which yields

$$\begin{aligned} H'_1 &= \frac{(8\pi\delta)^2}{2A^2} u^2 \delta + \frac{1}{2\pi} \frac{(8\pi\delta)^3}{A^3} \frac{A}{12} u^3 \cos(3\psi) \\ &= \frac{(8\pi)^2 \delta^3}{2A^2} u^2 + \frac{(8\pi)^2 \delta^3}{3A^2} u^3 \cos(3\psi) \\ &= \frac{(8\pi)^2 \delta^3}{3A^2} \left[u^3 \cos(3\psi) + \frac{3}{2} u^2 \right] . \end{aligned} \quad (27)$$

The next step is to put $u = u_0$ and $\cos(3\psi) = 1$ to obtain the value of the constant E :

$$E = \frac{(8\pi)^2 \delta^3}{3A^2} \left[u_0^3 + \frac{3}{2} u_0^2 \right] , \quad (28)$$

and then one substitutes these expressions into the decoherence formula Eq. (17), i.e.

$$\begin{aligned} k^2 E &= \frac{2A^2}{(12\pi)^2} \frac{(8\pi)^2 \delta^3}{3A^2} \left[u_0^3 + \frac{3}{2} u_0^2 \right] \\ &= \left(\frac{2}{3} \right)^3 \delta^3 \left[u_0^3 + \frac{3}{2} u_0^2 \right] , \end{aligned} \quad (29)$$

and so, from Eq. (17),

$$\begin{aligned} N &= \frac{2\delta^2}{3k^2 E} \\ &= \frac{2}{3} \delta^2 \frac{1}{\left(\frac{2}{3} \right)^3 \delta^3 \left[u_0^3 + \frac{3}{2} u_0^2 \right]} \\ &= \frac{1}{\left(\frac{2}{3} \right)^2 \delta \left[u_0^3 + \frac{3}{2} u_0^2 \right]} \end{aligned} \quad (30)$$

In Ref. [3], the values used were $\delta = 0.07$ and use $u_0 = 0.3$ and $u_0 = 0.4$ for the two values corresponding to $E = E_1$ and $E = E_2$, respectively. Let us put $\delta = 0.07$ and $u_0 = 0.35$. The result is $N = 141.8$ turns, as compared to $N = 141$ from Ref. [3], so the two derivations agree.

5 Differential Equation Method

In the Edwards method, I actually obtained $\theta = \theta(\psi)$, which was then inverted to find $\psi = \psi(\theta)$, to obtain the tunesift. Now let us solve Eqs. (2) and (3) to obtain $\psi = \psi(\theta)$ directly. As before, I put $I = I_0 + I_1 + \dots$, and also write $\psi = \psi_0 + \psi_1 + \psi_2 + \dots$, where the expansion parameter is k . The first approximation, using Eq. (2), is $\psi_0 = \delta\theta$. Recall also that $I_0 = E/\delta$. Then, from Eq. (3),

$$\begin{aligned}\frac{dI}{d\theta} &\simeq 3kI_0^{3/2} \sin(3\psi_0) = 3kI_0^{3/2} \sin(3\delta\theta) \\ I &= I_0 - \frac{kI_0^{3/2}}{\delta} \cos(3\delta\theta) \\ &= \frac{E}{\delta} - \frac{k}{\delta} \left(\frac{E}{\delta}\right)^{3/2} \cos(3\delta\theta),\end{aligned}\tag{31}$$

and so

$$I_1 = -\frac{k}{\delta} \left(\frac{E}{\delta}\right)^{3/2} \cos(3\delta\theta).\tag{32}$$

Returning to Eq. (3),

$$\begin{aligned}\frac{d}{d\theta}(\psi_0 + \psi_1) &= \delta + \frac{3}{2}kI_0^{1/2} \cos(3\psi_0) \\ &= \delta + \frac{3}{2}k \left(\frac{E}{\delta}\right)^{1/2} \cos(3\delta\theta),\end{aligned}\tag{33}$$

which yields

$$\begin{aligned}\frac{d\psi_1}{d\theta} &= \frac{3}{2}k \left(\frac{E}{\delta}\right)^{1/2} \cos(3\delta\theta) \\ \psi_1 &= \frac{k}{2\delta} \left(\frac{E}{\delta}\right)^{1/2} \sin(3\delta\theta).\end{aligned}\tag{34}$$

The next approximation is given by

$$\begin{aligned}\frac{d}{d\theta}(\psi_0 + \psi_1 + \psi_2) &\simeq \delta + \frac{3}{2}k(I_0 + I_1)^{1/2} \cos(3(\psi_0 + \psi_1)) \\ &\simeq \delta + \frac{3}{2}kI_0^{1/2} \left(1 + \frac{1}{2}\frac{I_1}{I_0}\right) (\cos(3\psi_0) - 3\psi_1 \sin(3\psi_0)),\end{aligned}\tag{35}$$

from which I find

$$\frac{d\psi_2}{d\theta} = \frac{3}{4}kI_0^{-1/2}I_1 \cos(3\psi_0) - \frac{3}{2}kI_0^{1/2} \sin(3\psi_0)3\psi_1$$

$$\begin{aligned}
&= -\frac{3}{4}k \left(\frac{E}{\delta}\right)^{-1/2} \frac{k}{\delta} \left(\frac{E}{\delta}\right)^{3/2} \cos^2(3\delta\theta) - \frac{3}{2}k \left(\frac{E}{\delta}\right)^{1/2} \frac{3k}{2\delta} \left(\frac{E}{\delta}\right)^{1/2} \sin^2(3\delta\theta) \\
&= -\frac{3}{4} \frac{k^2 E}{\delta^2} \cos^2(3\delta\theta) - \frac{9}{4} \frac{k^2 E}{\delta^2} \sin^2(3\delta\theta) \\
&= -\frac{3}{2} \frac{k^2 E}{\delta^2} + \text{oscillatory terms} .
\end{aligned} \tag{36}$$

Therefore, neglecting the oscillatory terms,

$$\delta' = \frac{d\psi}{d\theta} \simeq \delta - \frac{3}{2} \frac{k^2 E}{\delta^2} . \tag{37}$$

Hence, from Eq. (5),

$$N = \frac{1}{|\delta - \delta'|} = \frac{2\delta^2}{3k^2 E} , \tag{38}$$

which is the same as Eq. (17).

6 Canonical Transform Method

Here one seeks to perform a canonical transformation to diagonalize the Hamiltonian to first order in k . The transformed Hamiltonian will then yield an action-dependent tune, from which we can calculate the decoherence time. Let the final action-angle variables be $\{J, \Psi\}$. We see from Eq. (2) that $dI/d\theta = O(k)$; hence we want $dJ/d\theta = O(k^2)$. A suitable generating function is

$$F = J\psi - \frac{kJ^{3/2}}{3\delta} \sin(3\psi) . \tag{39}$$

Then

$$\begin{aligned}
I &= \frac{\partial F}{\partial \psi} = J - \frac{kJ^{3/2}}{\delta} \cos(3\psi) \\
\Psi &= \frac{\partial F}{\partial J} = \psi - \frac{kJ^{1/2}}{2\delta} \sin(3\psi) .
\end{aligned} \tag{40}$$

The transformed Hamiltonian is

$$\begin{aligned}
K &= \frac{\partial F}{\partial \theta} + H \\
&= \delta I + kI^{3/2} \cos(3\psi)
\end{aligned}$$

$$\begin{aligned}
&= \delta J - kJ^{3/2} \cos(3\psi) + k \left(J - \frac{kJ^{3/2}}{\delta} \cos(3\psi) \right)^{3/2} \cos(3\psi) \\
&= \delta J - \frac{3k^2 J^2}{2\delta} \cos^2(3\psi) + O(k^3) \\
&= \delta J - \frac{3k^2 J^2}{2\delta} \cos^2(3\Psi) + O(k^3) .
\end{aligned} \tag{41}$$

Thus $dJ/d\theta = -\partial K/\partial\Psi = O(k^2)$, as required. The new tune, to $O(k^2)$, is

$$\begin{aligned}
\delta' = \frac{d\Psi}{d\theta} = \frac{\partial K}{\partial J} &\simeq \delta - \frac{3k^2 J}{\delta} \cos^2(3\Psi) \\
&= \delta - \frac{3k^2 J}{2\delta} + \text{oscillatory terms} .
\end{aligned} \tag{42}$$

This agrees with all the previous formulas for the tunes shift. Using Eq. (5), with $J = E/\delta$ because we want $|\delta - \delta'|$ only to $O(k^2)$, the decoherence time is

$$N = \frac{1}{|\delta - \delta'|} = \frac{2\delta^2}{3k^2 E} , \tag{43}$$

in agreement with Eq. (17).

7 Lie Algebra: Deprit's/Michelotti's Method

A disadvantage of the standard canonical transformation technique is that the generating function depends on both the new and old dynamical variables. It is not a function of the old variables only. The use of Lie transformations gets around this problem. One such method is called Deprit's algorithm [4]. I shall actually employ the algorithm as described by Michelotti [5]. The perturbation expansion parameter is k , and functions H_n , K_n and S_n are needed,

$$H = \sum_{n=0}^{\infty} \frac{k^n}{n!} H_n , \quad K = \sum_{n=0}^{\infty} \frac{k^n}{n!} K_n , \quad S = \sum_{n=0}^{\infty} \frac{k^n}{n!} S_n . \tag{44}$$

Here H is the old Hamiltonian, K is the new (diagonalized) Hamiltonian, and S is the generator of the Lie transformation [5]. Obviously

$$H_0 = I\delta , \quad H_1 = I^{3/2} \cos(3\psi) , \quad H_2 = \dots = 0 . \tag{45}$$

Then, by definition,

$$\left(\frac{\partial}{\partial\theta} + \delta \frac{\partial}{\partial\psi} \right) S_1 + K_1 = H_1 = I^{3/2} \cos(3\psi) . \tag{46}$$

By definition, K_1 is chosen to equal the constant part of the r.h.s., so $K_1 = 0$. Then the solution for S_1 is

$$S_1 = \frac{I^{3/2}}{3\delta} \sin(3\psi). \quad (47)$$

The constant of integration is chosen so that S_1 is periodic in θ . Again by definition,

$$\begin{aligned} \left(\frac{\partial}{\partial \theta} + \delta \frac{\partial}{\partial \psi} \right) S_2 + K_2 &= H_2 + \{H_1, S_1\} + \{K_1, S_1\} \\ &= 0 + \{I^{3/2} \cos(3\psi), \frac{I^{3/2}}{3\delta} \sin(3\psi)\} + 0 \\ &= -3I^{3/2} \sin(3\psi) \frac{I^{1/2}}{2\delta} \sin(3\psi) - \frac{3}{2} I^{1/2} \cos(3\psi) \frac{I^{3/2}}{\delta} \cos(3\psi) \\ &= -\frac{3I^2}{2\delta}. \end{aligned} \quad (48)$$

Since K_2 is defined to equal the constant part of the r.h.s., $K_2 = -3I^2/(2\delta)$, which is the whole r.h.s., and so $S_2 = 0$. Therefore, to $O(k^2)$, the new Hamiltonian is

$$\begin{aligned} K &= K_0 + kK_1 + \frac{k^2}{2} K_2 + \dots \\ &= I\delta - \frac{3k^2 I^2}{4\delta} + \dots \end{aligned} \quad (49)$$

which is the same as in the previous section, neglecting oscillatory terms. The new tune is

$$\delta' = \frac{\partial K}{\partial I} = \delta - \frac{3k^2 I}{2\delta}, \quad (50)$$

which agrees with all previous formulas for the tunes, and hence yields the same decoherence time.

8 Lie Algebra: Forest's/Yokoya's Method

Ref. [5] is not the only formalism to apply Lie algebra to accelerator physics. Another approach has been developed by Dragt [6]. Diagonalization procedures using Dragt's formalism have been developed by Forest [7] and Yokoya [8]. Strictly speaking, both of these authors diagonalize not the Hamiltonian but the S-matrix (the one-turn map). In this section, therefore, a variant of their formalisms will be used, rather than a direct copy of their work. The

Hamiltonian is

$$H = I\delta + kI^{3/2} \cos(3\psi) \equiv H_2 + H_3 + \underbrace{H_4 + \dots}_{=0} \quad (51)$$

where $H_n = O(I^{n/2})$. The diagonalized Hamiltonian is given by the Lie transformation

$$K = \dots e^{-:V_4:} e^{-:V_3:} H, \quad (52)$$

where $:V:$ denotes a Poisson Bracket operator,

$$:f:g \equiv \{f, g\}. \quad (53)$$

The functions V_n are chosen to diagonalize H up to $O(I^{n/2})$. We therefore only need V_3 for our purposes, since we want the tunes shift to $O(k^2)$. Adapting the results of Refs. [7] and [8], the solution for V_3 is

$$V_3 - \int_{-\infty}^{\theta} H_3 d\theta' = \frac{kI^{3/2}}{3\delta} \sin(3\psi). \quad (54)$$

Then

$$\begin{aligned} K &\simeq e^{-:V_3:} H \\ &= H_2 + H_3 - :V_3:(H_2 + H_3) + \frac{1}{2} :V_3::V_3:H_2 + \dots \\ &= I\delta + kI^{3/2} \cos(3\psi) - \frac{k}{3} :I^{3/2} \sin(3\psi): I - \frac{k^2}{3\delta} :I^{3/2} \sin(3\psi): I^{3/2} \cos(3\psi) \\ &\quad + \frac{k^2}{18\delta} :I^{3/2} \sin(3\psi)::I^{3/2} \sin(3\psi): I + \dots \\ &= I\delta + kI^{3/2} \cos(3\psi) - kI^{3/2} \cos(3\psi) \\ &\quad - \frac{k^2}{3\delta} \left[3I^{3/2} \cos(3\psi) \frac{3}{2} I^{1/2} \cos(3\psi) + \frac{3}{2} I^{1/2} \sin(3\psi) 3I^{3/2} \sin(3\psi) \right] \\ &\quad + \frac{k^2}{6\delta} :I^{3/2} \sin(3\psi): I^{3/2} \cos(3\psi) + \dots \\ &= I\delta - \frac{3k^2 I^2}{4\delta} + \dots \end{aligned} \quad (55)$$

which is the same expression as in previous sections, for the diagonalized Hamiltonian, and so it leads to the same tunes shift and decoherence time.

9 Another decoherence criterion

9.1 Small kick amplitude

The decoherence criterion used in this note is not unique. We could have set $|\psi_{nl} - \psi_0| = 1$, not 2π , as the decoherence criterion in Eq. (5). The decoherence of a kicked beam caused by a betatron tune spread in the beam is calculated in Ref. [1]. It is assumed that the nonlinear tune ν is related to the linear tune ν_0 by (they use ν 's rather than δ 's) [9]

$$\nu = \nu_0 - \mu a^2, \quad (56)$$

where $a = \sqrt{\beta\epsilon}/\sigma_\beta$. Here ϵ is the Courant-Snyder invariant,

$$\epsilon = \frac{x^2 + (\alpha x + \beta x')^2}{\beta} = 2I, \quad (57)$$

$\sigma_\beta = \beta\sigma$ is the transverse beam size, and β is the beta function at $\theta = 0$. Hence $\langle a^2 \rangle = 2$, the average being over the beam. For a kick to the beam centroid of peak amplitude $|\bar{x}(0)|_{pk}$, the beam centroid amplitude on the n^{th} turn is given by [1]

$$|\bar{x}(n)|_{pk} = A(n)|\bar{x}(0)|_{pk}. \quad (58)$$

This equation defines the decoherence factor A . If $|\bar{x}(0)|_{pk} = Z\sigma_\beta$ and $Z \ll 1$,

$$A(n) \simeq \frac{1}{1 + (Q_p n)^2}, \quad (59)$$

where $Q_p = 4\pi\mu$. Thus the decoherence rate (inverse number of turns) is

$$N^{-1} = Q_p = 4\pi\mu = 2\pi\langle\Delta\nu\rangle. \quad (60)$$

However, the decoherence rate, according to the criterion Eq. (5), is

$$N^{-1} = \langle\Delta\nu\rangle, \quad (61)$$

when averaged over the beam. Thus the decoherence time calculated in this note, averaged over the beam, is a factor 2π longer than that in Ref. [1]. This can be understood as follows. If, instead of Eq. (5), we ask that $|\psi_{nl} - \psi_0| = 1$, not 2π , then

$$\begin{aligned} |\psi_{nl} - \psi_0| &= 1 \\ |2\pi\delta'N - 2\pi\delta N| &= 1 \\ N^{-1} &= 2\pi|\delta - \delta'|, \end{aligned} \quad (62)$$

which yields Eq. (60) when averaged over the beam.

9.2 Large kick amplitude

The full solution for A in Ref. [1] is

$$A(n) = \frac{1}{1 + (Q_p n)^2} \exp \left[-\frac{1}{2} \frac{(Q_g n)^2}{1 + (Q_p n)^2} \right], \quad (63)$$

where $Q_g = ZQ_p = 2\pi Z\langle\Delta\nu\rangle$. For a large kick amplitude $Z \gg 1$, the decoherence factor can be approximated by

$$A(n) \simeq e^{-(Q_g n)^2/2}, \quad (64)$$

which is a Gaussian with a standard deviation of Q_g^{-1} . Hence the decoherence rate is

$$N^{-1} = Q_g = 2\pi Z\langle\Delta\nu\rangle, \quad (65)$$

and is a factor Z larger, or the decoherence time is a factor Z smaller, than the small kick amplitude value. This agrees with the heuristic analysis presented earlier for the decoherence criterion.

Note that factors of order unity have been ignored above, in fixing the value of the decoherence time. The decoherence criterion has *not* been fixed as a half-life, or $1/e$ life, of the beam centroid amplitude. Instead, qualitative measures of the time scales have been used. Therefore one can only say that the large kick amplitude decoherence time is of *order* Z smaller than the small kick amplitude value. One cannot pin down factors of 2, etc. this way.

9.3 Intermediate kick amplitude

If $Z \simeq 2$ or 3, then a Gaussian approximation for the decoherence factor $A(n)$ is less valid than if $Z = 5$ or 10. From the Gaussian approximation above, the beam centroid amplitude is a factor $e^{-1/2}$ of its initial value after one decoherence time. Using this criterion, we can calculate the correction to the decoherence time if Z does not greatly exceed unity:

$$\begin{aligned} \exp \left[-\frac{1}{2} \frac{(Q_g N)^2}{1 + (Q_p N)^2} \right] &\simeq e^{-1/2} \\ \frac{(Q_g N)^2}{1 + (Q_p N)^2} &= 1 \\ N^{-2} &= Q_g^2 - Q_p^2 \\ N &= \frac{1}{Q_g \sqrt{1 - Z^{-2}}}. \end{aligned} \quad (66)$$

The Lorentzian $[1 + (Q_p n)^2]^{-1}$ outside the exponential in Eq. (63) leads to a further logarithmic correction, which I shall ignore in this report. For $Z = 2 - 3$, the non-Gaussian nature of $A(n)$ leads to an increase of about 5 – 20% over the simple Gaussian approximation for the decoherence time.

10 Additional phase modulation

In practice the betatron phase, if modulated by sextupoles, varies according to

$$\frac{d\psi}{d\theta} = \delta_0 + \delta_1 a \cos(\lambda\psi) + \delta_2 a^2 + \dots \quad (67)$$

where λ might be 3, but we can consider arbitrary values. It is adequate to approximate $d\psi/d\theta = \delta_0$ in the cosine above. Note that the normalization is $\langle a^2 \rangle = 2$, hence $\delta_2 = -\mu$. It is assumed in the Ref. [1] derivation of $A(n)$ that the contribution of the δ_1 term is negligible. In this section I shall estimate the effect of such a term. I start with

$$x = \sqrt{2I\beta} \cos \left(\int \frac{d\psi}{d\theta'} d\theta' \right), \quad (68)$$

where the phase is given by

$$\int \frac{d\psi}{d\theta'} d\theta' = \psi_0 + (\delta_0 + \delta_2 a^2)\theta + \frac{\delta_1 a}{\lambda \delta_0} [\sin(\lambda\psi) - \sin(\lambda\psi_0)]. \quad (69)$$

The population density function is given by

$$\rho(a, \psi_0) da d\psi_0 = a e^{-(a^2 + Z^2 - 2aZ \cos \psi_0)/2} \frac{da d\psi_0}{2\pi}. \quad (70)$$

The beam centroid is now calculated as in Ref. [1]:

$$\begin{aligned} \bar{x} &= \sqrt{2\beta} \int \frac{da d\psi_0}{2\pi} a e^{-(a^2 + Z^2 - 2aZ \cos \psi_0)/2} \times \\ &\quad \times \text{Re} \left\{ e^{i\psi_0} e^{i[(\delta_0 + \delta_2 a^2)\theta + 2\delta_1 a(\lambda\delta_0)^{-1} \sin(\lambda(\psi_0 + \delta_0\theta/2)) \cos(\lambda\delta_0\theta/2)]} \right\} \\ &= \sqrt{2\beta} \int \frac{da d\psi_0}{2\pi} a e^{-(a^2 + Z^2)/2} \sum_m e^{im\psi_0} I_m(aZ) \times \\ &\quad \times \text{Re} \left\{ e^{i\psi_0} e^{i(\delta_0 + \delta_2 a^2)\theta} \sum_k e^{ik\lambda(\psi_0 + \delta_0\theta/2)} J_k \left(\frac{2\delta_1 a}{\lambda\delta_0} \cos \left(\frac{\lambda\delta_0\theta}{2} \right) \right) \right\}. \quad (71) \end{aligned}$$

The integral over ψ_0 kills all terms for which $m \pm (1 + k\lambda) \neq 0$, i.e. one must have $m = \pm(1 + k\lambda)$. Previously, $\delta_1 = 0$, and so $J_k = 0$ unless $k = 0$, hence $m = \pm 1$ only. Now there is an infinite sum. The result is

$$\bar{x} = \sqrt{2\beta} \int da a e^{-(a^2+Z^2)/2} \times \\ \times \text{Re} \left\{ \sum_k I_{1+k\lambda}(aZ) J_k \left(\frac{2\delta_1 a}{\lambda\delta_0} \cos \left(\frac{\lambda\delta_0 \theta}{2} \right) \right) e^{i(\delta_0 + \delta_2 a^2)\theta + ik\lambda\delta_0 \theta/2} \right\}. \quad (72)$$

Gradshteyn and Ryzik [11] have a solution for this as a sum of hypergeometric functions, but that merely trades one complicated expression for another. Note, however, that we basically want the Gaussian part of the decoherence factor, if we restrict attention to large amplitude kicks to the beam. This can be obtained by using the asymptotic expansions of the Bessel functions. For large arguments,

$$I_\nu(z) \approx \frac{e^z}{\sqrt{2\pi z}}, \quad J_\nu(z) \approx \frac{\sqrt{2} \cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}{\sqrt{\pi z}}. \quad (73)$$

To demonstrate, consider the $\delta_1 = 0$ case again. Then

$$\bar{x} \simeq \sqrt{2\beta} \int da a e^{-(a^2+Z^2)/2} I_1(aZ) e^{i(\delta_0 + \delta_2 a^2)\theta} \\ \simeq \sqrt{2\beta} e^{-Z^2/2} \int da a e^{-(1-i2\delta_2\theta)a^2/2} \frac{e^{aZ}}{\sqrt{2\pi aZ}}. \quad (74)$$

The exponent is a quadratic polynomial in a , and yields a Gaussian if I complete the square, which I therefore do, and obtain

$$(1 - i2\delta_2\theta) \frac{a^2}{2} - aZ = \frac{(1 - i2\delta_2\theta)}{2} \left[a - \frac{Z}{1 - i2\delta_2\theta} \right]^2 - \frac{Z^2}{2(1 - i2\delta_2\theta)} \quad (75)$$

and I pull the last term out of the integral, since it does not depend on a . This yields

$$\bar{x} \propto \sqrt{2\beta} e^{-Z^2/2} e^{Z^2/[2(1-i2\delta_2\theta)]} \times \text{integral of "order unity"} \\ \propto \exp \left[\frac{iZ^2\delta_2\theta}{1 - i2\delta_2\theta} \right] \\ \propto \exp \left[\frac{iZ^2\delta_2\theta(1 + i2\delta_2\theta)}{1 + 4\delta_2^2\theta^2} \right] \\ \propto \exp \left[-\frac{(2Z\delta_2\theta)^2}{2(1 + 4\delta_2^2\theta^2)} \right] \exp \left[\frac{iZ^2\delta_2\theta}{1 + 4\delta_2^2\theta^2} \right]$$

$$\Rightarrow A(\theta) \propto \exp \left[-\frac{1}{2} \frac{(2Z\delta_2\theta)^2}{1 + 4\delta_2^2\theta^2} \right] = \exp \left[-\frac{1}{2} \frac{(Q_g n)^2}{1 + (Q_p n)^2} \right], \quad (76)$$

which is the exponential part of the decoherence factor in Ref. [1] or Eq. (63).

Let us now consider $\delta_1 \neq 0$. Now we need to include the J Bessel function as well. Just as for the I Bessel function, the contribution to the exponential behavior of \bar{x} is determined by the large argument approximation. Let us define

$$K = \frac{2\delta_1}{\lambda\delta_0} \cos \left(\frac{\lambda\delta_0\theta}{2} \right) \quad (77)$$

for convenience. Then

$$\begin{aligned} \bar{x} &\simeq \sqrt{2\beta} \int da \, a e^{-(a^2+Z^2)/2} e^{i\delta_2 a^2 \theta} \frac{e^{aZ}}{\sqrt{2\pi aZ}} \frac{e^{iaK}}{\sqrt{2\pi aK}} \\ &\propto e^{-Z^2/2} \int da \, a e^{-(1-i2\delta_2\theta)a^2/2} e^{a(Z+iK)} \\ &\propto e^{-Z^2/2} e^{(Z+iK)^2/[2(1-i2\delta_2\theta)]}. \end{aligned} \quad (78)$$

In the above derivation I have ignored the sum over k , and various complex exponentials which do not contribute to the exponential part of the decoherence factor, although of course they contribute to the phase of \bar{x} . The exponential part of the decoherence factor is just the magnitude of the above expression, viz.

$$\begin{aligned} A(\theta) &\propto e^{-Z^2/2} \exp \left[\frac{Z^2 - K^2}{2(1 + 4\delta_2^2\theta^2)} \right] \exp \left[-\frac{2ZK\delta_2\theta}{1 + 4\delta_2^2\theta^2} \right] \\ &= \exp \left[-\frac{1}{2} \frac{(2Z\delta_2\theta + K)^2}{1 + 4\delta_2^2\theta^2} \right]. \end{aligned} \quad (79)$$

It has the same Gaussian part as before but with a shifted origin. Since K is bounded, because $|K| \leq 2\delta_1/(\lambda\delta_0)$, the shift of the origin is irrelevant if

$$2|Z\delta_2\theta| \gg |K|. \quad (80)$$

Therefore we can ignore δ_1 and continue to use the previous decoherence formula if we make a cut on small values of θ given by

$$\theta \gg \left| \frac{K}{2Z\delta_2} \right| = \left| \frac{\delta_1}{Z\lambda\delta_0\delta_2} \right| \quad (81)$$

or, in terms of a number of turns,

$$N_{cut} = \frac{\theta}{2\pi} \gg \left| \frac{\delta_1}{2\pi Z\lambda\delta_0\delta_2} \right|. \quad (82)$$

Since $N_{decoh} = (4\pi|\delta_2|)^{-1}$ from Ref. [1] for $Z \gg 1$, we can rewrite this result in the form

$$N_{cut} \gg \left| \frac{2\delta_1}{\lambda\delta_0} \right| N_{decoh} . \quad (83)$$

However, we must also have $N_{cut} \ll N_{decoh}$, or else there will be region of useful data, hence it is necessary that

$$\left| \frac{2\delta_1}{\lambda\delta_0} \right| \ll 1 . \quad (84)$$

However, in order to extract the correction to the Gaussian part of the exponential, I used a large argument approximation to the J Bessel function. This means $|K| \gg m$ for J_m , in particular $|K| \gg 1$, i.e.

$$\left| \frac{2\delta_1}{\lambda\delta_0} \right| \gg 1 . \quad (85)$$

Therefore, if $|K| \gg 1$, there is no region of useful data to which a Gaussian decoherence factor as in Ref. [1] can be fit. On the other hand, if $|K| \ll 1$, then the decoherence factor of Ref. [1] is satisfactory. The J Bessel functions then add a correction only to the non-exponential part of the decoherence factor.

The conclusion, in words rather than equations, is that δ_0 should not be smaller than δ_1 , i.e. a particle must not be so close to a resonance that $d\psi/d\theta$ crosses the resonant tune as it oscillates. The beam as a whole should be sufficiently far off resonance so that a “1- σ ” particle does not cross the resonant tune. This fact is actually pointed out in Ref. [1] (although a mathematical bound is not given), because the authors state that their calculation assumes the nonlinear distortion of the phase-space trajectories is small, basically only a tuneshift, and this is valid only far off resonance.

11 Discrete-sextupole Hamiltonian

In the above calculations a generalized sextupole strength k was used. To compare with experimental results, I need to express the Hamiltonian in terms of the strengths $B''/(B\rho)$ of the actual sextupoles in a ring. I follow the notation of Merminga and Ng [10], with modifications to conform to the notation already used above. The Hamiltonian is

$$H = \nu I + \frac{RB''}{6B\rho} x^3 . \quad (86)$$

Here R is the average machine radius and

$$x = \sqrt{2I\beta_x} \cos(\psi + \Psi - \nu\theta) . \quad (87)$$

The linear action-angle variables are I and ψ , respectively, and $\Psi = \int \beta_x^{-1} ds$ is the Floquet phase. Now define $Q = \Psi - \nu\theta$ and

$$s_k = \left(\frac{\beta_x^3}{\beta_0} \right)_k^{1/2} \frac{(B''L)_k}{2B\rho}, \quad (88)$$

where β_0 is a constant, and L is the length of the k^{th} sextupole. Then

$$\begin{aligned} \frac{RB''}{6B\rho} \left(\frac{\beta_x}{\beta_0} \right)^{3/2} x^3 &= (2I)^{3/2} \frac{RB''}{6B\rho} \beta_x^{3/2} \cos^3(\psi + Q) \\ &= \frac{(2I)^{3/2} \beta_0^{1/2}}{4} \frac{RB''}{6B\rho} \left(\frac{\beta_x}{\beta_0} \right)^{1/2} [\cos[3(\psi + Q)] + 3\cos(\psi + Q)] \\ &= \frac{(2I)^{3/2} \beta_0^{1/2}}{8} \frac{RB''}{6B\rho} \left(\frac{\beta_x}{\beta_0} \right)^{1/2} [e^{i3(\psi+Q)} + 3e^{i(\psi+Q)} + \text{c.c.}] . \end{aligned} \quad (89)$$

Since the functions

$$\frac{RB''}{B\rho} \left(\frac{\beta_x}{\beta_0} \right)^{1/2} e^{\pm i3Q}$$

are periodic in θ , I can expand them into harmonics, viz.

$$\frac{1}{48} \frac{RB''}{B\rho} \left(\frac{\beta_x}{\beta_0} \right)^{1/2} e^{\pm i3Q} \equiv \sum_{m=-\infty}^{\infty} A_{\pm 3m} e^{i\alpha_{\pm 3m}} e^{-im\theta} \quad (90)$$

where $A_{\pm 3m}$ and $\alpha_{\pm 3m}$ are real. Inverting the Fourier series,

$$\begin{aligned} A_{\pm 3m} e^{i\alpha_{\pm 3m}} &= \frac{1}{2\pi} \oint d\theta \frac{RB''}{48B\rho} \left(\frac{\beta_x}{\beta_0} \right)^{1/2} e^{\pm i3Q} e^{im\theta} \\ &= \frac{1}{48\pi} \sum_k \Delta\theta_k \left(\frac{RB''}{2B\rho} \left(\frac{\beta_x}{\beta_0} \right)^{1/2} \right)_k e^{\pm i3Q_k} e^{im\theta_k} \quad \left(\Delta\theta_k = \frac{L_k}{R} \right) \\ &= \frac{1}{48\pi} \sum_k s_k e^{\pm i3Q_k} e^{im\theta_k} . \end{aligned} \quad (91)$$

From this one can easily show that

$$\begin{aligned} A_{-3m} &= A_{3-m} \\ \alpha_{-3m} &= -\alpha_{3-m} . \end{aligned} \quad (92)$$

The Hamiltonian can now be written as

$$H = \nu I + (2I)^{3/2} \beta_0^{1/2} \sum_m \left[A_{3m} e^{i(3\psi + \alpha_{3m} - m\theta)} + A_{-3m} e^{-i(3\psi - \alpha_{-3m} + m\theta)} \right] , \quad (93)$$

where only terms resonant at third-integer values have been retained. I perform a canonical transformation to change the linear tune to $\delta = \nu - (n/3)$, where $n/3$ is the nearest third-integer harmonic to ν . Let the new action-angle variables be $\{J, \psi'\}$. The generating function is

$$F = J(\psi - \frac{n}{3}\theta) , \quad (94)$$

and so

$$\begin{aligned} I &= \frac{\partial F}{\partial \psi} = J \\ \psi' &= \frac{\partial F}{\partial J} = \psi - \frac{n}{3}\theta \\ K &= H + \frac{\partial F}{\partial \theta} = H - \frac{n}{3}J . \end{aligned} \quad (95)$$

Thus

$$K = \delta J + (2J)^{3/2} \beta_0^{1/2} \sum_m \left[A_{3m} e^{i(3\psi' + \alpha_{3m} - (m-n)\theta)} + A_{-3m} e^{i(3\psi' - \alpha_{-3m} + (m+n)\theta)} \right] . \quad (96)$$

To avoid proliferation of notation, I shall rename the Hamiltonian H and call the action-angle variables I and ψ again. To compare with the calculations in the previous sections, I now retain only the harmonic closest to the resonance. In the first exponential this is given by $m - n = 0$ and in the second exponential I select the term with $m + n = 0$. Then

$$\begin{aligned} H &= \delta I + (2I)^{3/2} \beta_0^{1/2} \left[A_{3n} e^{i(3\psi + \alpha_{3n})} + A_{-3-n} e^{i(3\psi - \alpha_{-3-n})} \right] \\ &= \delta I + 2^{5/2} I^{3/2} \beta_0^{1/2} A_{3n} \cos[3(\psi + \alpha_{3n})] , \end{aligned} \quad (97)$$

using the relations between A_{3-n} and A_{-3n} , and α_{3-n} and α_{-3n} . The above Hamiltonian can now be easily compared to that in the previous sections. One can identify the generalized sextupole strength k as

$$k = 2^{5/2} \beta_0^{1/2} A_{3n} \quad (98)$$

and so the expression for the tunesht is

$$\begin{aligned} \delta' - \delta &= -\frac{3k^2 I}{2\delta} \\ &= -\frac{48\beta_0 I}{\delta} A_{3n}^2 \end{aligned}$$

$$= -\frac{48\beta_0 I}{\delta} \frac{1}{(48\pi)^2} \left| \sum_k s_k e^{i(n\theta_k + 3Q_k)} \right|^2 \quad (99)$$

The above expression makes it clear that the tuneshift is negative definite, for an arbitrary sextupole distribution, as has been long known, and thus provides a check on the calculation. Expanding in a double sum,

$$\delta' - \delta = -\frac{\beta_0 I}{48\pi^2 \delta} \sum_{k,k'} s_k s_{k'} \exp[i(n(\theta_k - \theta_{k'}) + 3(Q_k - Q_{k'}))] . \quad (100)$$

The decoherence time is of course given by $N = |\delta - \delta'|^{-1}$. Since the tuneshift is real, the above expression can be replaced by its real part, hence

$$N^{-1} = \frac{\beta_0 I}{48\pi^2 \delta} \sum_{k,k'} s_k s_{k'} \cos[n(\theta_k - \theta_{k'}) + 3(Q_k - Q_{k'})] , \quad (101)$$

which avoids the use of complex numbers in numerical evaluations. The tuneshift formula is a special case of those of Collins [12] and Ohnuma [13], which have been shown to be equivalent by Ng [14]. These authors include all the harmonics, including first-integer as well as third-integer terms (or the equivalent in their respective notations).

12 Conclusions

I have shown how to calculate the decoherence time, for a specific model, using several different formalisms [3] – [8]. I have shown that they all yield the same answer, having first specified a well-defined model-independent criterion for the decoherence time. I have also treated various modifications to the assumptions of the basic theory, e.g. large kick amplitudes or phase lags of 1 rather than 2π radians, and derived the resulting changes to the decoherence time.

Detailed mathematical derivations of the decoherence factor, hence the decoherence time, are given in Ref. [1]. The former contains an exponential factor, which is approximately a Gaussian for large kick amplitudes (i.e. kick amplitudes much larger than the r.m.s. transverse beam size). I have supplied various heuristic derivations for both the decoherence time and the exponential part of the decoherence factor, and in addition given an approximate mathematical bound for the validity of some of the approximations made in Ref. [1]. For the model treated in Ref. [1], if the r.m.s. tunespread of the beam is $\langle \Delta\nu \rangle$, and the initial kick amplitude is $Z\sigma_\beta$, where σ_β is the r.m.s. transverse beam size due to betatron motion, then if $Z \ll 1$ the decoherence time is

$$N_{\text{decoh}} \simeq \frac{1}{\langle \Delta\nu \rangle} \quad (Z \ll 1) , \quad (102)$$

while if $Z \gg 1$ then the decoherence time is

$$N_{decoh} \simeq \frac{1}{Z\langle\Delta\nu\rangle} \quad (Z \gg 1), \quad (103)$$

but if Z is only slightly larger than unity, then I find a correction

$$N_{decoh} \simeq \frac{1}{Z\langle\Delta\nu\rangle} \frac{1}{\sqrt{1-Z^{-2}}} \quad (Z > 1, Z \not\gg 1), \quad (104)$$

which increases the decoherence time by about 5 – 20% for $Z \simeq 2 - 3$. *Note that the decoherence criterion for the small amplitude kick case is actually different from the latter two cases (where the criterion is the same), because it is based on a Lorentzian not a Gaussian.*

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